Technology
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# Efficient Quantum Algorithms for Stabilizer Entropies

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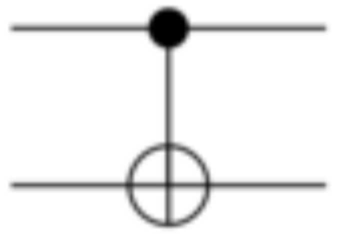
IMPERIAL

# Nonstabilizerness (magic)

- Stabilizer states and Clifford operations can be efficiently simulated
- Universal quantum gate set:

*Clifford operations*



$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$


$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$


$$= \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

**+**

*(non-Clifford) T-gates*


$$= \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

- **Nonstabilizerness** characterise the amount of non-Clifford resources needed to realise quantum states and operations [1,2]

1. T. Haug and M.S. Kim. ‘Scalable Measures of Magic Resource for Quantum Computers’. PRX Quantum 4, 010301 (2023)
2. L. Leone, S. F. E. Oliviero, and A. Hamma. ‘Stabilizer Rényi Entropy’. Physical Review Letters 128, 050402 (2022)

# Stabilizer Entropy (SE)

- Related to various properties of quantum systems
  - Phase transitions, entanglement spectrum, property testing, participation entropy, etc.

**Phase transition in magic with random quantum circuits**

Pradeep Niroula,[1,2,*] Christopher David White,[1] Qingfeng Wang,[3,2] Sonika Johri,[4]
Daiwei Zhu,[4] Christopher Monroe,[1,2,5,4] Crystal Noel,[5] and Michael J. Gullans[1,†]
[1]*Joint Center for Quantum Information and Computer Science, University of Maryland and NIST, College Park, MD 20742*
[2]*Joint Quantum Institute, University of Maryland and NIST, College Park, MD 20742*
[3]*Chemical Physics Program and Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742*
[4]*IonQ, Inc., College Park, MD 20740*
[5]*Duke Quantum Center, Department of Electrical & Computer Engineering and Department of Physics, Duke University, Durham, NC 27701*

**Phase transition in Stabilizer Entropy and efficient purity estimation**

Lorenzo Leone,[1,*] Salvatore F.E. Oliviero,[1,†] Gianluca Esposito,[2,‡] and Alioscia Hamma[2,3,§]
[1]*Physics Department, University of Massachusetts Boston, 02125, USA*
[2]*Dipartimento di Fisica 'Ettore Pancini', Università degli Studi di Napoli Federico II, Via Cintia 80126, Napoli, Italy*
[3]*INFN, Sezione di Napoli, Italy*

**Nonstabilizerness determining the hardness of direct fidelity estimation**

Lorenzo Leone,[1,*] Salvatore F.E. Oliviero,[1,†] and Alioscia Hamma[2,3,‡]
[1]*Physics Department, University of Massachusetts Boston, 02125, USA*
[2]*Dipartimento di Fisica 'Ettore Pancini', Università degli Studi di Napoli Federico II, Via Cintia 80126, Napoli, Italy*
[3]*INFN, Sezione di Napoli, Complesso universitario di Monte S.Angelo ed 6 via Cintia, 80126, Napoli, Italy*

**Quantifying non-stabilizerness through entanglement spectrum flatness**

Emanuele Tirrito,[1,2] Poetri Sonya Tarabunga,[1,2,3] Gugliemo Lami,[2] Titas Chanda,[1,2] Lorenzo Leone,[4] Salvatore F.E. Oliviero,[4] Marcello Dalmonte,[1,2] Mario Collura,[2,3] and Alioscia Hamma[5,6]
[1]*The Abdus Salam International Centre for Theoretical Physics (ICTP), Strada Costiera 11, 34151 Trieste, Italy*
[2]*SISSA, Via Bonomea 265, 34136 Trieste, Italy*
[3]*INFN, Sezione di Trieste, Via Valerio 2, 34127 Trieste, Italy*
[4]*Physics Department, University of Massachusetts Boston, 02125, USA*
[5]*Dipartimento di Fisica 'Ettore Pancini', Università degli Studi di Napoli Federico II, Via Cintia 80126, Napoli, Italy*
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# Stabilizer Entropy (SE)

- Related to various properties of quantum systems
  - Phase transitions, entanglement spectrum, property testing, participation entropy, etc.
  - Connection to quantum chaos and scrambling (out-of-time ordered correlators, OTOCs)

**Shannon and entanglement entropies of one- and two-dimensional critical wave functions**

Jean-Marie Stéphan,[1] Shunsuke Furukawa,[2] Grégoire Misguich,[1] and Vincent Pasquier[1]

[1] *Institut de Physique Théorique, CEA, IPhT, CNRS, URA 2306, F-91191 Gif-sur-Yvette, France.*
[2] *Condensed Matter Theory Laboratory, RIKEN, Wako, Saitama 351-0198, Japan*
(Dated: 05/10/2009)

**Resource theory of quantum scrambling**

Roy J. Garcia,[1,*] Kaifeng Bu,[1,†] and Arthur Jaffe[1,‡]

[1] *Harvard University, Cambridge, Massachusetts 02138, USA*
(Dated: October 6, 2022)

Quantum Chaos is Quantum

Lorenzo Leone[1], Salvatore F.E. Oliviero[1], You Zhou[2,3], and Alioscia Hamma[1]

[1]Physics Department, University of Massachusetts Boston, 02125, USA
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[3]Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA

# Stabilizer Entropy (SE)

- Related to various properties of quantum systems
  - Phase transitions, entanglement spectrum, property testing, participation entropy, etc.
  - Connection to quantum chaos and scrambling (OTOCs)
- Experimental studies limited by measurement protocols which scale exponentially with number of qubits [1,2]

1. T. Haug and M.S. Kim. 'Scalable Measures of Magic Resource for Quantum Computers'. PRX Quantum 4, 010301 (2023)
2.S. F. E. Oliviero, L. Leone, A. Hamma, and S. Lloyd, "Measuring magic on a quantum processor," npj Quantum Information 8, 148 (2022)

# Stabilizer Entropy (SE)

- Related to various properties of quantum systems
  - Phase transitions, entanglement spectrum, property testing, participation entropy, etc.
  - Connection to quantum chaos and scrambling (OTOCs)
- Experimental studies limited by measurement protocols which scale exponentially with number of qubits [1,2]
- Efficient algorithms exist for matrix product states

**Quantifying Nonstabilizerness of Matrix Product States**

Tobias Haug[1,*] and Lorenzo Piroli[2]
[1] *QOLS, Blackett Laboratory, Imperial College London SW7 2AZ, UK*
[2] *Philippe Meyer Institute, Physics Department, École Normale Supérieure (ENS), Université PSL, 24 rue Lhomond, F-75231 Paris, France*
(Dated: January 31, 2023)

**Stabilizer entropies and nonstabilizerness monotones**

Tobias Haug[1] and Lorenzo Piroli[2]
[1]QOLS, Blackett Laboratory, Imperial College London SW7 2AZ, UK
[2]Philippe Meyer Institute, Physics Department, École Normale Supérieure (ENS), Université PSL, 24 rue Lhomond, F-75231 Paris, France
August 22, 2023

# Renyi-$n$ SE[1]

- For an $N$-qubit state $|\psi\rangle$,

$$M_n(|\psi\rangle) = -(1-n)^{-1}\ln \sum_{\sigma \in \mathscr{P}} \frac{\langle\psi|\sigma|\psi\rangle^{2n}}{2^N}$$

- Where $n$ is the index of SE and $\mathscr{P}$ is set of $4^N$ Pauli strings
- Pauli strings are $N$-qubit tensor products $\sigma_r = \bigotimes_{j=1}^{N} \sigma_{r_{2j-1}r_{2j}}$ where $r \in \{0,1\}^{2N}$

*Pauli matrices*

$\sigma_{00} = I_1 \quad \sigma_{01} = \sigma^x$

$\sigma_{10} = \sigma^z \quad \sigma_{11} = \sigma^y$

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$$M_n\left(|\psi_{STAB}\rangle\right) = 0 \qquad M_n\left(U_C|\psi_{STAB}\rangle\right) = M_n\left(|\psi_{STAB}\rangle\right)$$
$$M_n\left(|\psi\rangle \otimes |\phi\rangle\right) = M_n\left(|\psi\rangle\right) + M_n\left(|\phi\rangle\right)$$

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# Measuring SE

- For an $N$-qubit state $|\psi\rangle$,

$$M_n(|\psi\rangle) = -(1-n)^{-1} \ln \sum_{\sigma \in \mathcal{P}} \frac{\langle\psi|\sigma|\psi\rangle^{2n}}{2^N}$$

$$A_n(|\psi\rangle) = \sum_{\sigma \in \mathcal{P}} \frac{\langle\psi|\sigma|\psi\rangle^{2n}}{2^N}$$

$4^N$

# Measuring SE

- For an $N$-qubit state $|\psi\rangle$,

$$M_n(|\psi\rangle) = -(1-n)^{-1} \ln \underbrace{\sum_{\sigma \in \mathcal{P}} \frac{\langle \psi | \sigma | \psi \rangle^{2n}}{2^N}}$$

$$A_n(|\psi\rangle) = \sum_{\sigma \in \mathcal{P}} \frac{\langle \psi | \sigma | \psi \rangle^{2n}}{2^N}$$

- **—> How can we efficiently measure $A_n(|\psi\rangle)$ ?**

# Our paper

- **Efficiently measure SEs with integer index $n > 1$ on quantum computers and simulators**

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  - Efficient protocol to measure $4n$*-point OTOCs* and *Clifford-averaged multifractal flatness*

# Our paper

- **Efficiently measure SEs with integer index $n > 1$ on quantum computers and simulators**
- We provide:
  - Algorithm via Bell measurements over two copies of state
  - Efficient protocol to measure $4n$-*point OTOCs* and *Clifford-averaged multifractal flatness*
  - Efficiently computable bounds to other nonstabilizerness monotones

# Algorithm 1 for odd $n > 1$

*$n$: index of SE*
*$N$: no. of qubits*

$$M_n(|\psi\rangle) = -(1-n)^{-1}\ln A_n(|\psi\rangle)$$

- Consider $A_n$ as the expectation value of observable $\Gamma_n^{\otimes N}$ acting on $2n$ copies of $|\psi\rangle$ via the replica trick[1]

$$A_n = 2^{-N}\sum_{\sigma\in\mathcal{P}}\langle\psi|\sigma|\psi\rangle^{2n}$$
$$= 2^{-N}\sum_{\sigma\in\mathcal{P}}\langle\psi|^{\otimes 2n}\sigma^{\otimes 2n}|\psi\rangle^{\otimes 2n} = \langle\psi|^{\otimes 2n}\Gamma_n^{\otimes N}|\psi\rangle^{\otimes 2n}$$

- Where $\Gamma_n = \frac{1}{2}\sum_{k=0}^{3}(\sigma^k)^{\otimes 2n}$

1. J. M. Stéphan, S. Furukawa, G. Misguich, and V. Pasquier, Phys. Rev. B 80, 184421 (2009).

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**For odd** $n > 1$, $2^{-N}A_n$ is unitary with $\omega \in \{-1,1\}$

**For even** $n > 1$, $2^{-N}A_n$ is projector with $\omega \in \{0,2^N\}$

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# Algorithm 1 for odd $n > 1$

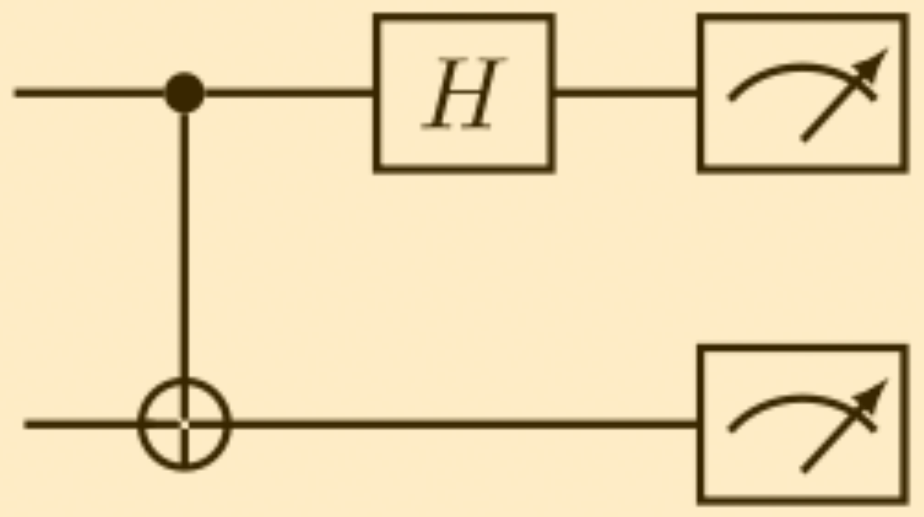
- To measure $\Gamma_n^{\otimes N}$, transform the operator into a diagonal eigenbasis

$$
\begin{aligned}
A_n \;&= \langle\psi|^{\otimes 2n} \left(U_{\mathrm{Bell}}^{\otimes n\,\dagger} \frac{1}{2}\left((I_1 \otimes I_1)^{\otimes n} + (\sigma^z \otimes I_1)^{\otimes n}\right.\right. \\
&\left.\left. + (I_1 \otimes \sigma^z)^{\otimes n} + (-1)^n (\sigma^z \otimes \sigma^z)^{\otimes n}\right) U_{\mathrm{Bell}}^{\otimes n}\right)^{\otimes N} |\psi\rangle^{\otimes 2n}
\end{aligned}
$$

$$U_{Bell} = (H \otimes I_1)CNOT$$

$$H = \frac{1}{\sqrt{2}}(\sigma^x + \sigma^z)$$

$$CNOT = \exp\left(i\frac{\pi}{4}(I_1 - \sigma^z) \otimes (I_1 - \sigma^x)\right)$$




Bell measurement

# Algorithm 1 for odd $n > 1$

- To measure $\Gamma_n^{\otimes N}$, transform the operator into a diagonal eigenbasis

$$A_n = \langle\psi|^{\otimes 2n} \left(U_{\text{Bell}}^{\otimes n}{}^{\dagger} \frac{1}{2}((I_1 \otimes I_1)^{\otimes n} + (\sigma^z \otimes I_1)^{\otimes n} + (I_1 \otimes \sigma^z)^{\otimes n} + (-1)^n (\sigma^z \otimes \sigma^z)^{\otimes n}) U_{\text{Bell}}^{\otimes n}\right)^{\otimes N} |\psi\rangle^{\otimes 2n}$$

$$= \left(U^{\otimes N}|\psi\rangle \otimes |\psi\rangle\right)^{\otimes n}$$

- **Sufficient to prepare and measure two copies of $N$-qubit quantum states, on a $2N$-qubit quantum computer simultaneously**

$$|\eta\rangle = U_{\text{Bell}}^{\otimes N} |\psi\rangle \otimes |\psi\rangle$$

# Algorithm 1 for odd $n > 1$

- To measure $\Gamma_n^{\otimes N}$, transform the operator into a diagonal eigenbasis

$$A_n = \langle\psi|^{\otimes 2n} (U_{\text{Bell}}^{\otimes n}{}^{\dagger} \frac{1}{2}((I_1 \otimes I_1)^{\otimes n} + (\sigma^z \otimes I_1)^{\otimes n} + (I_1 \otimes \sigma^z)^{\otimes n} + (-1)^n (\sigma^z \otimes \sigma^z)^{\otimes n}) \underbrace{U_{\text{Bell}}^{\otimes n})^{\otimes N} |\psi\rangle^{\otimes 2n}}_{= \left(U^{\otimes N}|\psi\rangle \otimes |\psi\rangle\right)^{\otimes n}}$$

- **Sufficient to prepare and measure two copies of $N$-qubit quantum states, on a $2N$-qubit quantum computer simultaneously**

- $A_n$ computed via post-processing of Bell measurement outcomes

# Algorithm 1 for odd $n > 1$

**Algorithm 1:** SE without complex conjugate

**Input :** Integer $n > 1$; $L$ repetitions
State preparation routine for $|\psi\rangle$

**Output:** Tsallis SE $T_n(|\psi\rangle)$

$A = 0$
**for** $\ell = 1, \dots, L$ **do**
  **for** $j = 1, \dots, n$ **do**
    Prepare $|\eta\rangle = U_{\text{Bell}}^{\otimes N} |\psi\rangle \otimes |\psi\rangle$
    Sample in computational basis $\boldsymbol{r}^{(j)} \sim |\langle \boldsymbol{r}|\eta\rangle|^2$
  **end**
  $b = 1$
  **for** $\ell = 1, \dots, N$ **do**
    $\nu_1 = \bigoplus_{j=1}^{n} r_{2\ell-1}^{(j)}$; $\nu_2 = \bigoplus_{j=1}^{n} r_{2\ell}^{(j)}$
    **if** $n$ is odd **then**
      $b = b \cdot (-2\nu_1 \cdot \nu_2 + 1)$
    **else**
      $b = b \cdot 2(\nu_1 - 1) \cdot (\nu_2 - 1)$
    **end**
  **end**
  $A = A + b/L$
**end**
$T_n = -(1-n)^{-1}(1-A)$

# Maximal number of $L$ measurement steps

- To estimate $A_n$ within $\epsilon$ accuracy and $\delta$ failure probability, we require at most

$$L \leq \frac{\Delta\omega_n^2}{2\epsilon^2} \log(\frac{2}{\delta})$$

- Where $\Delta\omega_n$ is the range of eigenvalues of $\Gamma_n^{\otimes N}$

**For odd** $n > 1, \omega \in \{-1,1\}$

**For even** $n > 1, \ \omega \in \{0,2^N\}$

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**For odd** $n > 1$, $\omega \in \{-1,1\}$ → **Odd**: $\Delta\omega_n = 2$ and $C = O(n\epsilon^{-2})$

**For even** $n > 1$, $\omega \in \{0,2^N\}$ → **Even**: $\Delta\omega_n$ diverges and requires exponential number of $L$

# Algorithm 2 for any integer $n > 1$

$$A_n(|\psi\rangle) = \sum_{\sigma \in \mathcal{P}} \frac{\langle \psi | \sigma | \psi \rangle^{2n}}{2^N}$$

- Requires access to complex conjugate $|\psi^*\rangle$
- Rewrite SE as a sampling problem: $A_n = \mathbb{E}_{\sigma \sim \Xi(\sigma)} [\langle \psi | \sigma | \psi \rangle^{2n-2}]$
  - Where $\Xi(\sigma) = 2^{-N} \langle \psi | \sigma | \psi \rangle^2$ is the probability distribution of Pauli strings $\sigma$

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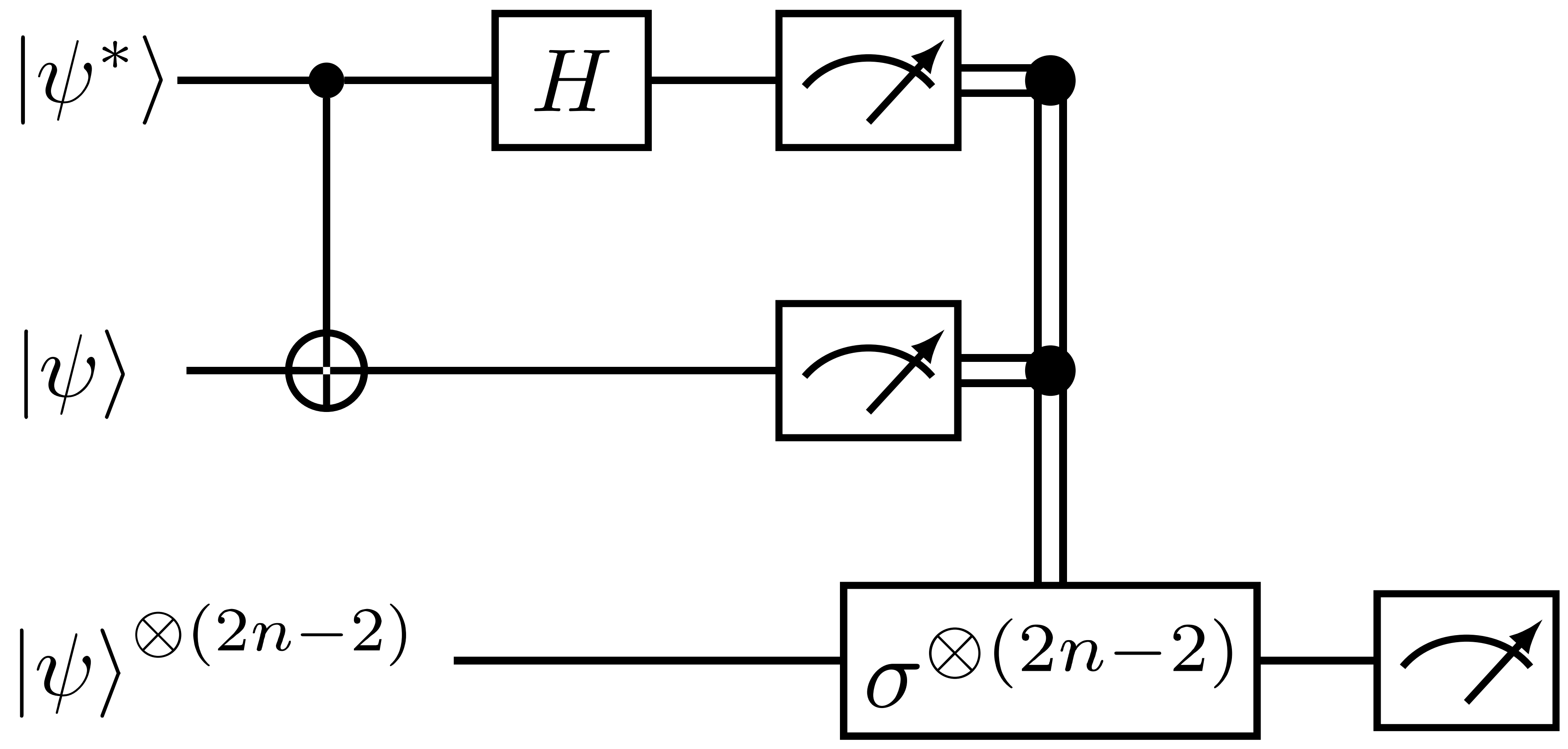

Algorithm 2 measurement protocol

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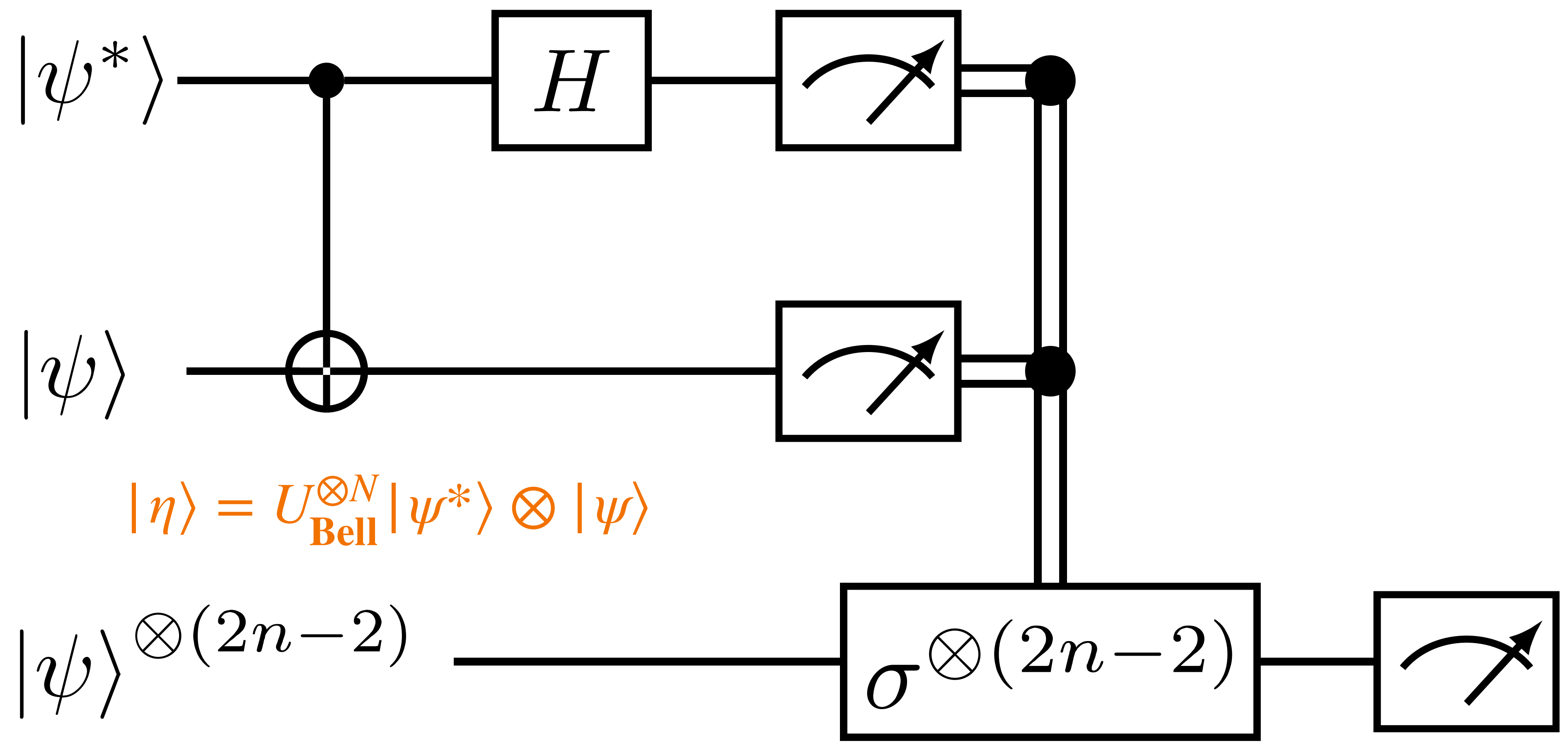


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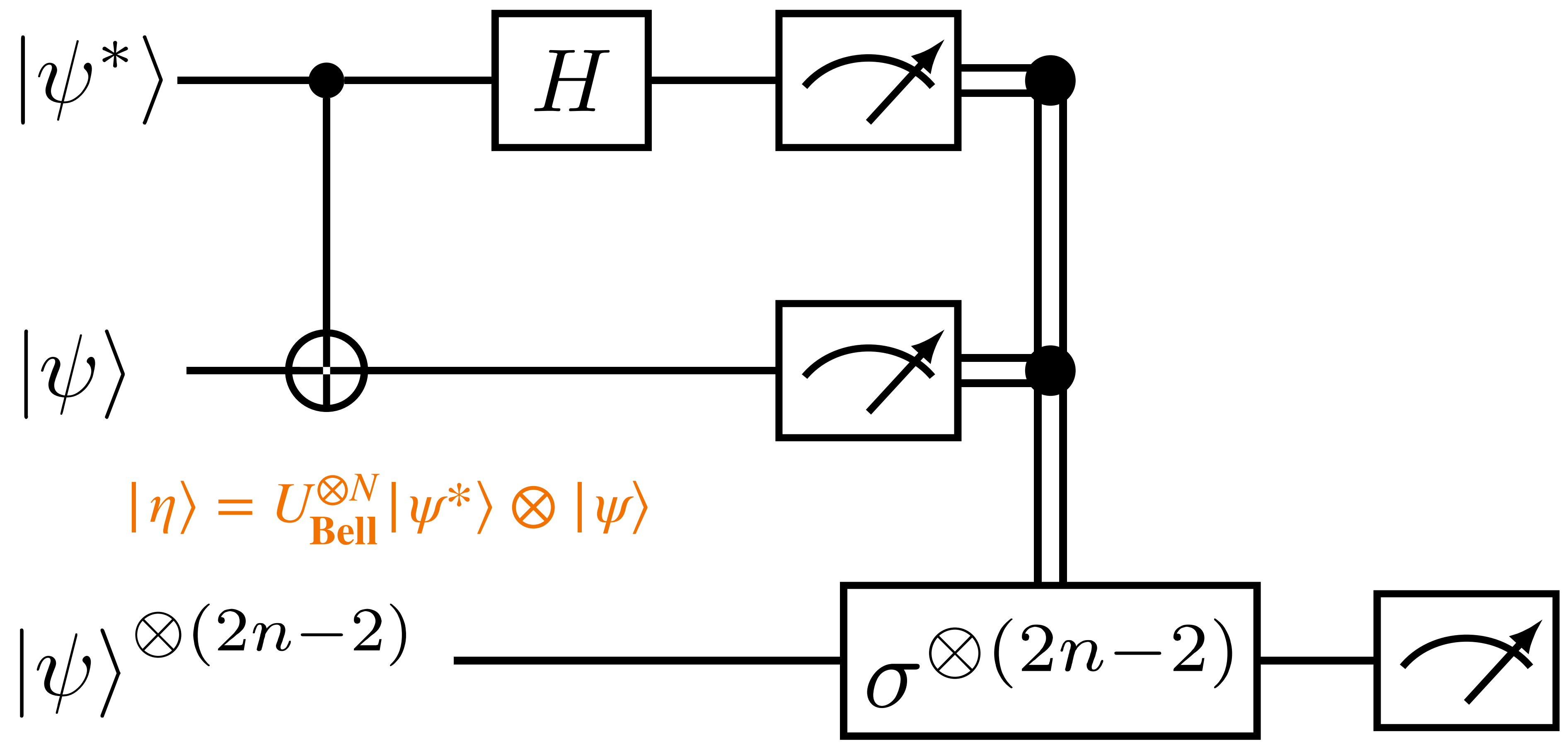


Algorithm 2 measurement protocol

- Gaining outcome $\mathbf{r} \in \{0,1\}^{2N}$

$$A_n(|\psi\rangle) = \sum_{\sigma \in \mathcal{P}} \frac{\langle\psi|\sigma|\psi\rangle^{2n}}{2^N}$$

# Algorithm 2 for any integer $n > 1$

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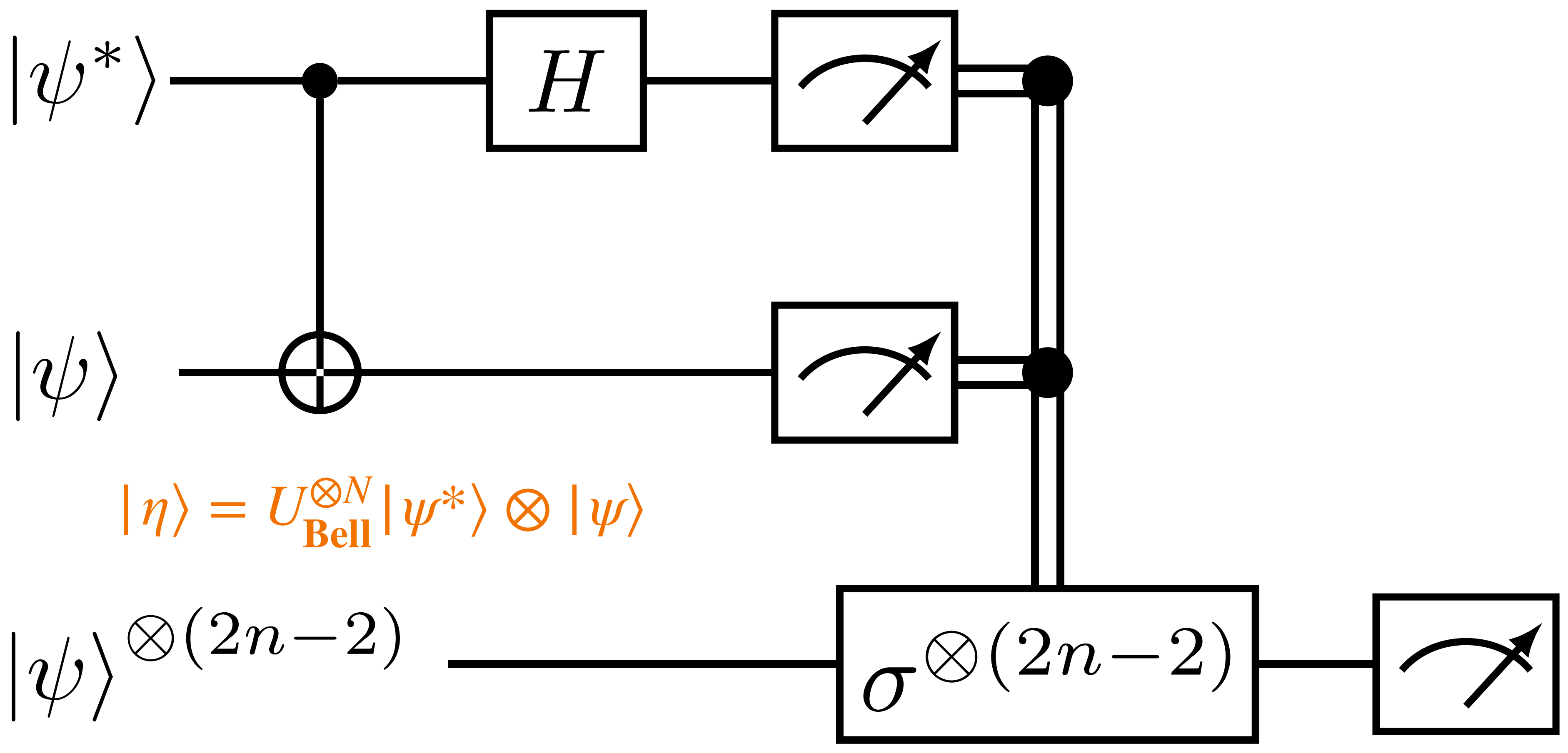


Algorithm 2 measurement protocol

- Gaining outcome $\mathbf{r} \in \{0,1\}^{2N}$
- $\Xi(\sigma_\mathbf{r}) = |\langle\mathbf{r}|\eta\rangle|^2$
- Sampling $\mathbf{r}$ from $|\eta\rangle$ corresponds to sampling Pauli strings $\sigma_\mathbf{r} \sim \Xi(\sigma_\mathbf{r})$

# Algorithm 2 for any integer $n > 1$

- Require at most $C = O(\alpha\epsilon^{-2})$ copies of $|\psi\rangle$ and $C = O(\epsilon^{-2})$ copies of $|\psi^*\rangle$

**Algorithm 2:** SE with complex conjugate

**Input :** Integer $n > 1$; $L$ repetitions;
State preparation routines for $|\psi\rangle$ and $|\psi^*\rangle$

**Output:** $A_n(|\psi\rangle)$

$A_n = 0$
**for** $\ell = 1, \dots, L$ **do**
  Prepare $|\eta\rangle = U_{\mathrm{Bell}}^{\otimes N} |\psi^*\rangle \otimes |\psi\rangle$
  Sample $\boldsymbol{r} \sim |\langle \boldsymbol{r}|\eta\rangle|^2$
  $b = 1$
  **for** $k = 1, \dots, 2n-2$ **do**
    Prepare $|\psi\rangle$ and measure in eigenbasis of Paulistring $\sigma_{\boldsymbol{r}}$ for eigenvalue $\lambda \in \{+1, -1\}$
    $b = b \cdot \lambda$
  **end**
  $A_n = A_n + b/L$
**end**


1. Y. Yang, G. Chiribella, and G. Adesso, Physical Review A 90, 042319 (2014)
2. J. Miyazaki, A. Soeda, and M. Murao, Physical Review Research 1, 013007 (2019)
3. T. Haug, K. Bharti, and D. E. Koh, arXiv:2306.11677 (2023)
4. S. Khatri, R. LaRose, A. Poremba, L. Cincio, A. T. Sornborger, and P. J. Coles, Quantum 3, 140 (2019)

# Algorithm 2 for any integer $n > 1$

- Require at most $C = O(\alpha\epsilon^{-2})$ copies of $|\psi\rangle$ and $C = O(\epsilon^{-2})$ copies of $|\psi^*\rangle$
- $|\psi^*\rangle$ cannot be efficiently prepared in general with black-box access to $|\psi\rangle$ [1–3]
- When the circuit description of $U$ preparing the state is known, $|\psi^*\rangle$ is constructed by an element-wise conjugation of coefficient of $U$ [4]

**Algorithm 2:** SE with complex conjugate

**Input :** Integer $n > 1$; $L$ repetitions;
State preparation routines for $|\psi\rangle$ and $|\psi^*\rangle$

**Output:** $A_n(|\psi\rangle)$

```
A_n = 0
for ℓ = 1, ..., L do
    Prepare |η⟩ = U_Bell^{⊗N} |ψ*⟩ ⊗ |ψ⟩
    Sample r ~ |⟨r|η⟩|^2
    b = 1
    for k = 1, ..., 2n − 2 do
        Prepare |ψ⟩ and measure in eigenbasis of
          Paulistring σ_r for eigenvalue λ ∈ {+1, −1}
        b = b · λ
    end
    A_n = A_n + b/L
end
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# Bounds on nonstabilizerness

- Number of stabilizer states scale as $O(2^{N^2})$, other nonstabilizerness monotones ($R, \xi, F_{STAB}$) become numerically infeasible to compute beyond a few qubits[2]

- Our algorithm provides efficient bounds for integer $n > 1$

$$R \geq \xi \geq F_{\mathrm{STAB}}^{-1} \geq A_n^{-\frac{1}{2n}}$$

- We also prove lower bound on $F_{STAB}$ for $n > 1$:

$$A_n^{\frac{1}{2n}} \geq F_{\mathrm{STAB}} \geq \frac{A_n - 2^{1-n}}{1 - 2^{1-n}}$$

$R$: Robustness of magic[2]

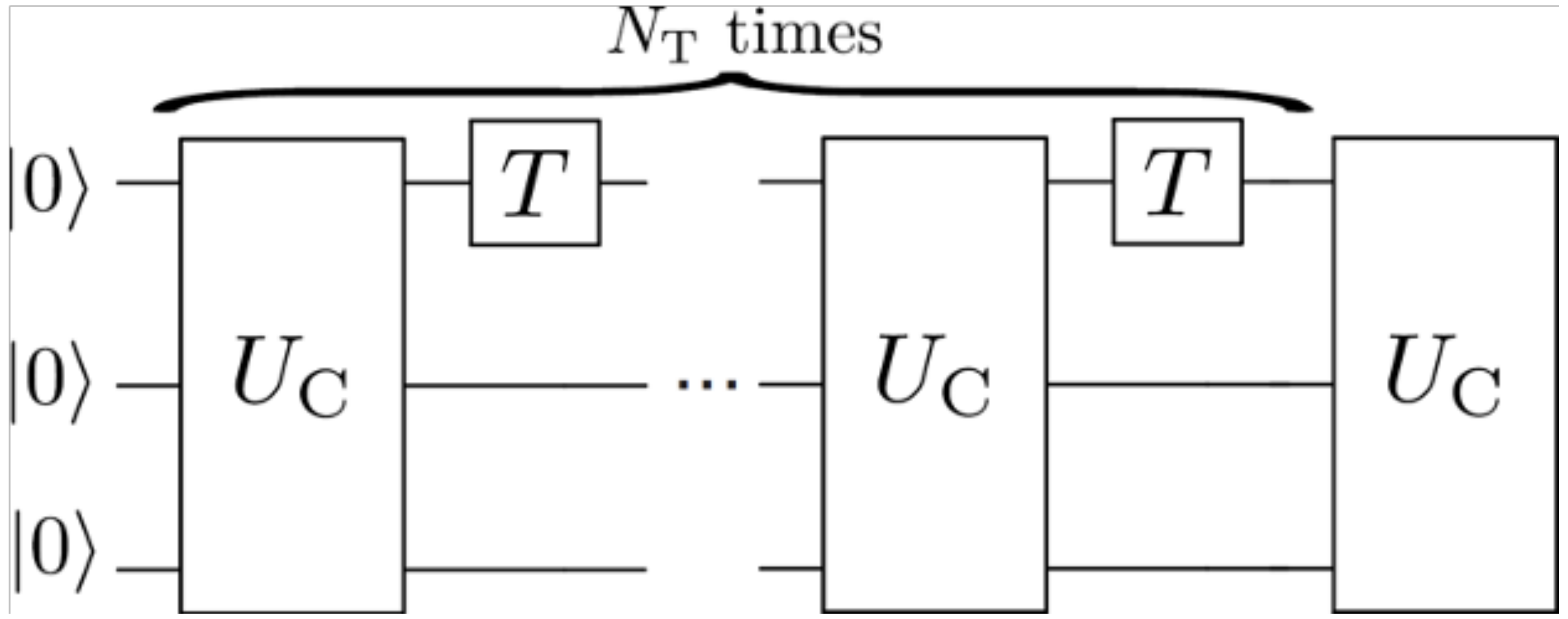
$\xi$: Stabilizer extent[1]

$F_{STAB} = \max_{|\phi\rangle \in STAB} |\langle \psi | \phi \rangle|^2$: Stabilizer fidelity[1]


1. S. Bravyi, D. Browne, P. Calpin, E. Campbell, D. Gosset, and M. Howard, Quantum 3, 181 (2019)
2. M. Howard and E. Campbell, Phys. Rev. Lett. 118, 090501 (2017)

# Demonstration using IonQ

- Random $U_C$ doped with $N_T$ non-Clifford gates



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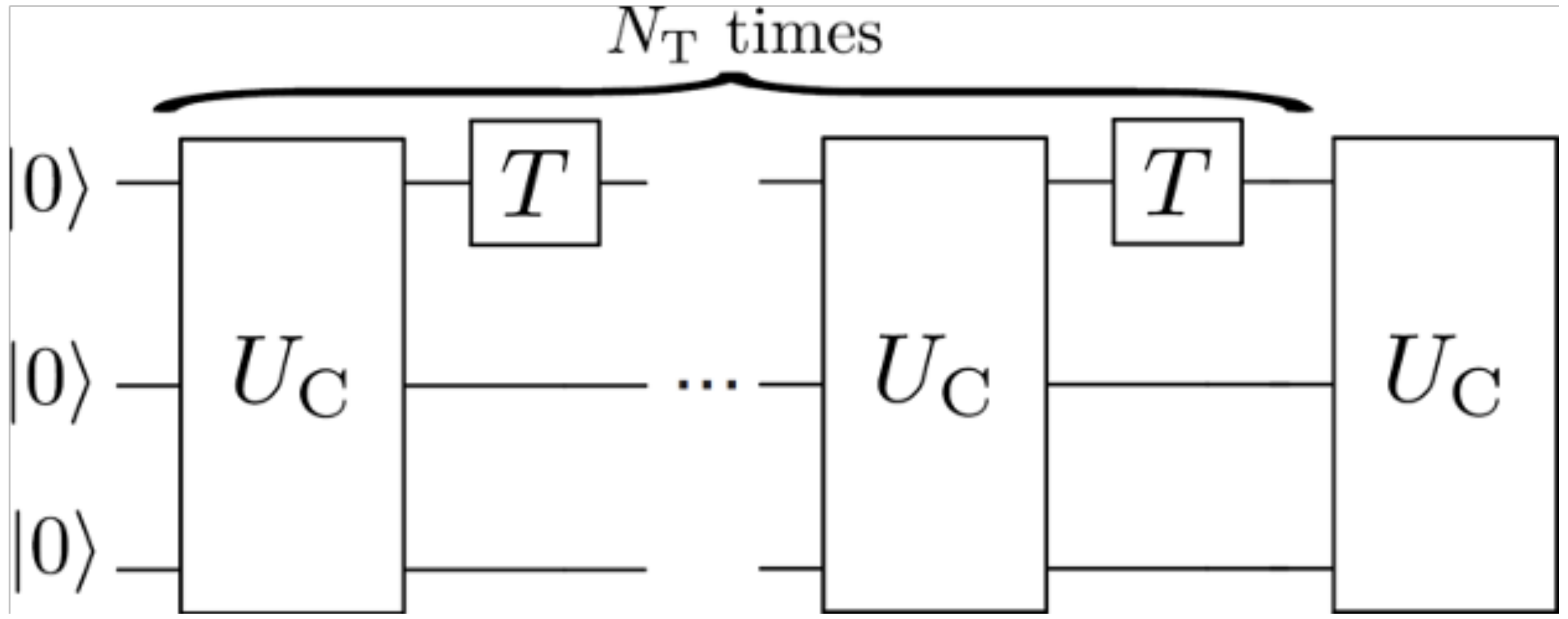


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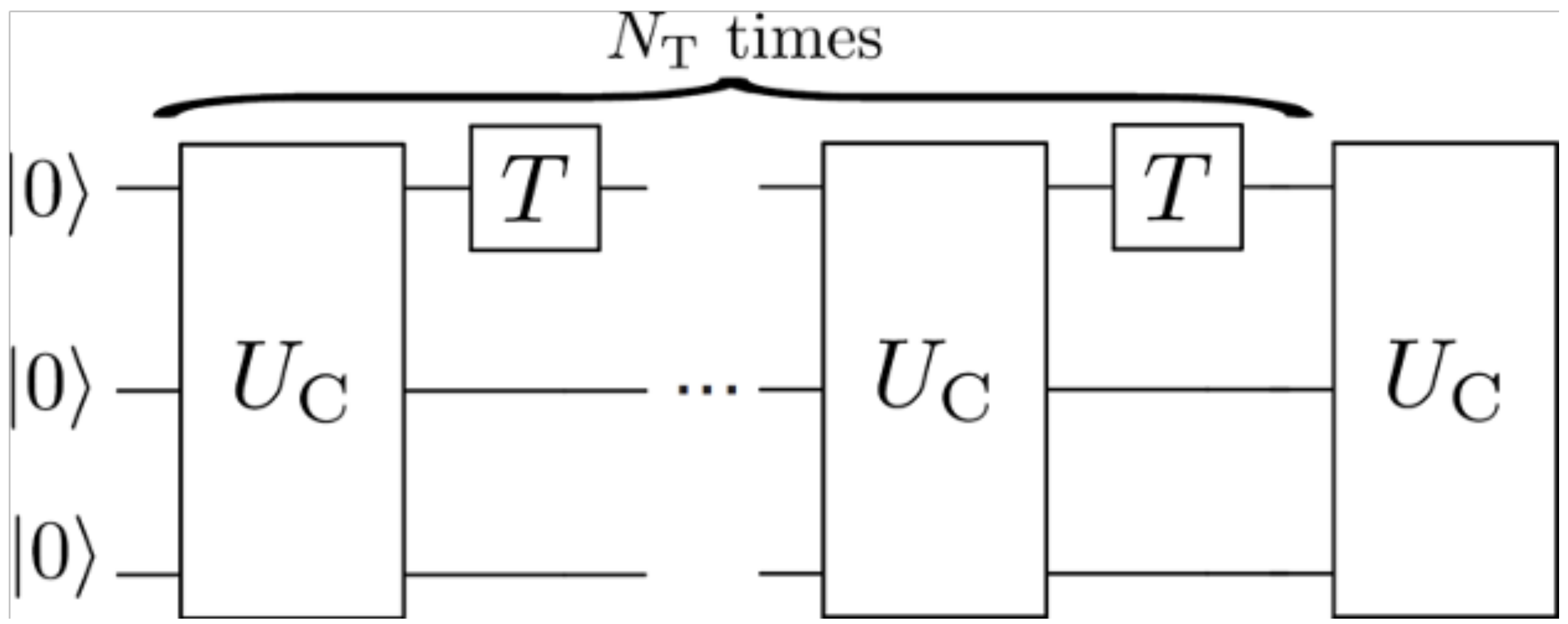


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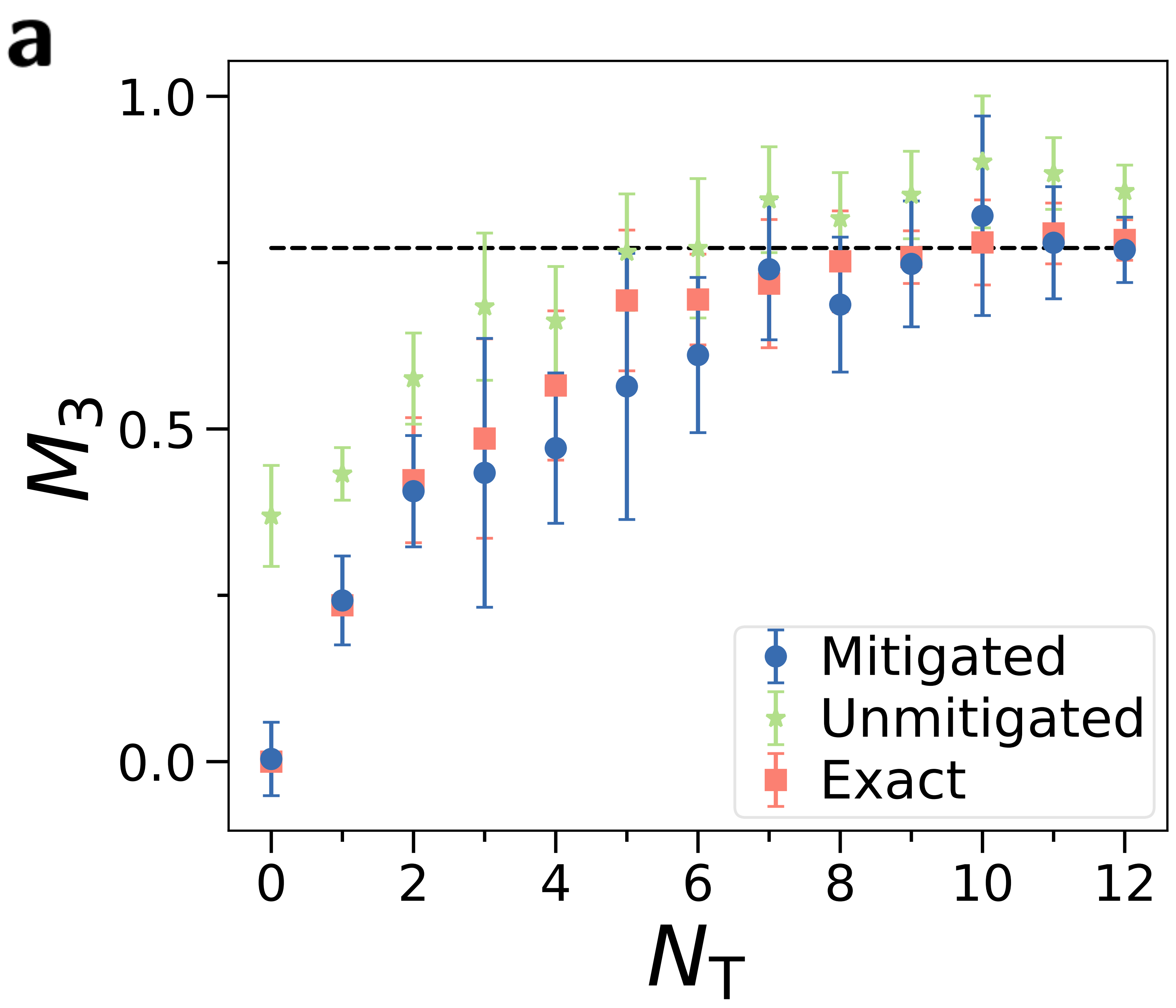

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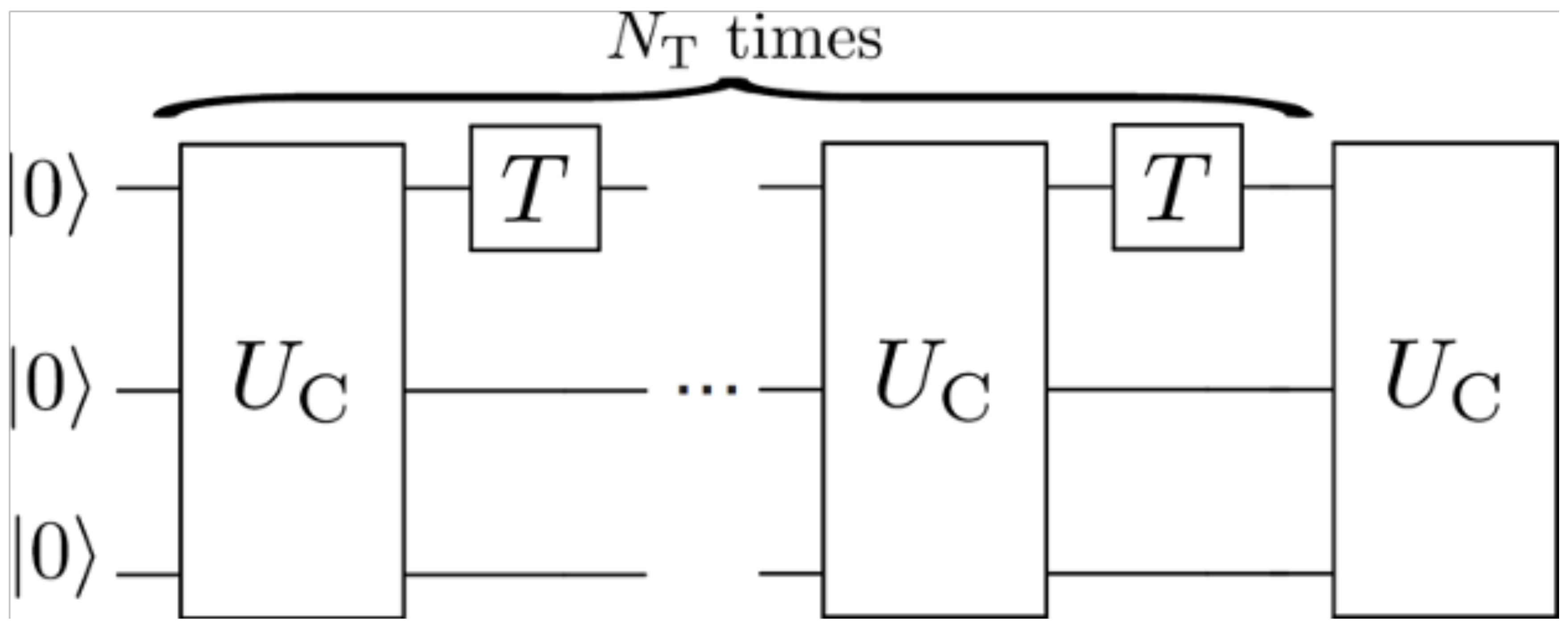


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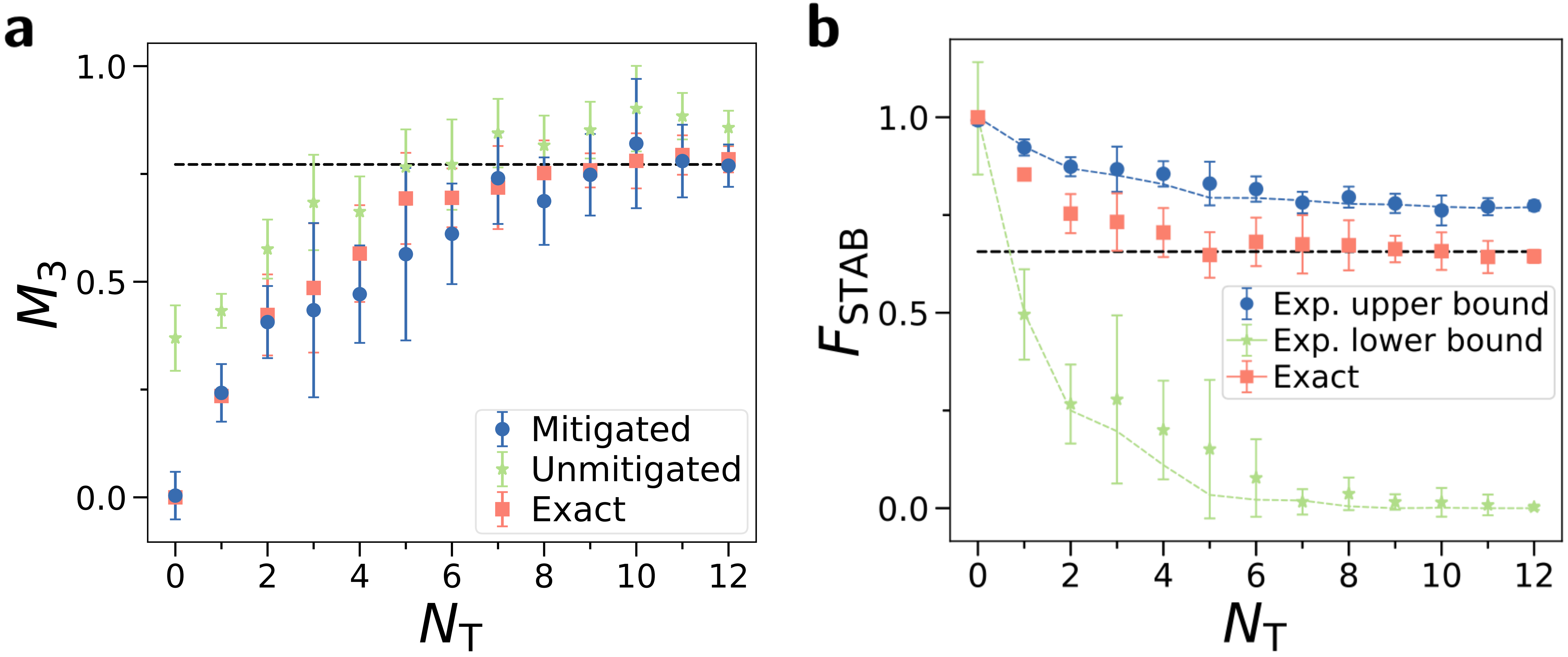


$$A_n^{\frac{1}{2n}} \geq F_{\text{STAB}} \geq \frac{A_n - 2^{1-n}}{1 - 2^{1-n}}$$

# Application 1. Clifford-averaged OTOCs

- Efficiently measure $4n$-point OTOC[1] averaged over all Pauli-strings for $N$-qubit unitaries $U$ and $n > 1$

1. L. Leone, S. F. E. Oliviero, and A. Hamma, Phys. Rev. A 107, 022429 (2023)

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$$\mathrm{otoc}_{4n}(U,\sigma,\sigma') = \left(2^{-N}\mathrm{tr}(\sigma U \sigma' U^\dagger)\right)^{2n}$$
$$= 2^{-N}\mathrm{tr}\Big(\langle \sigma^{(2n)} \prod_{i=1}^{2n} (U\sigma U^\dagger \sigma' \sigma^{(i-1)}\sigma^{(i)})\rangle_{\sigma^{(1)},\ldots,\sigma^{(2n)}}\Big)$$

→

$$\mathrm{OTOC}_{4n}(U) = 4^{-N}\sum_{\sigma,\sigma'\in\mathcal{P}} \mathrm{otoc}_{4n}(U,\sigma,\sigma')$$

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- Using Choi state $|U\rangle = I_N \otimes U \,|\Phi\rangle$ , where $|\Phi\rangle = 2^{-N/2} \sum_{i=0}^{2^N-1} |i\rangle \otimes |i\rangle$

$$\mathop{\mathbb{E}}_{U_\mathrm{C}, U'_\mathrm{C} \in \mathcal{C}_N} \left[\mathrm{otoc}_{4n}(U_\mathrm{C} U U'_\mathrm{C}, \sigma, \sigma')\right] = \frac{A_n(|U\rangle) 4^N - 1}{(4^N - 1)^2}$$

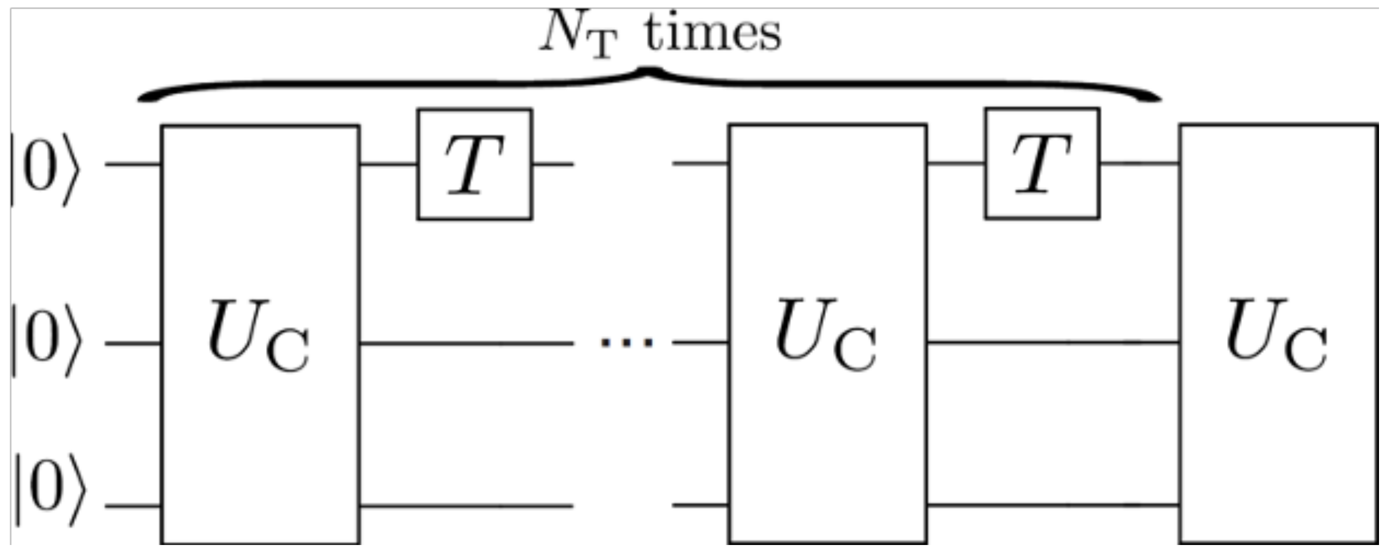
1. L. Leone, S. F. E. Oliviero, and A. Hamma, Phys. Rev. A 107, 022429 (2023)
S. Khatri, R. LaRose, A. Poremba, L. Cincio, A. Sornborger, and P. J Coles, Quantum 3, 140 (2019)

# Application 2. Multifractal flatness

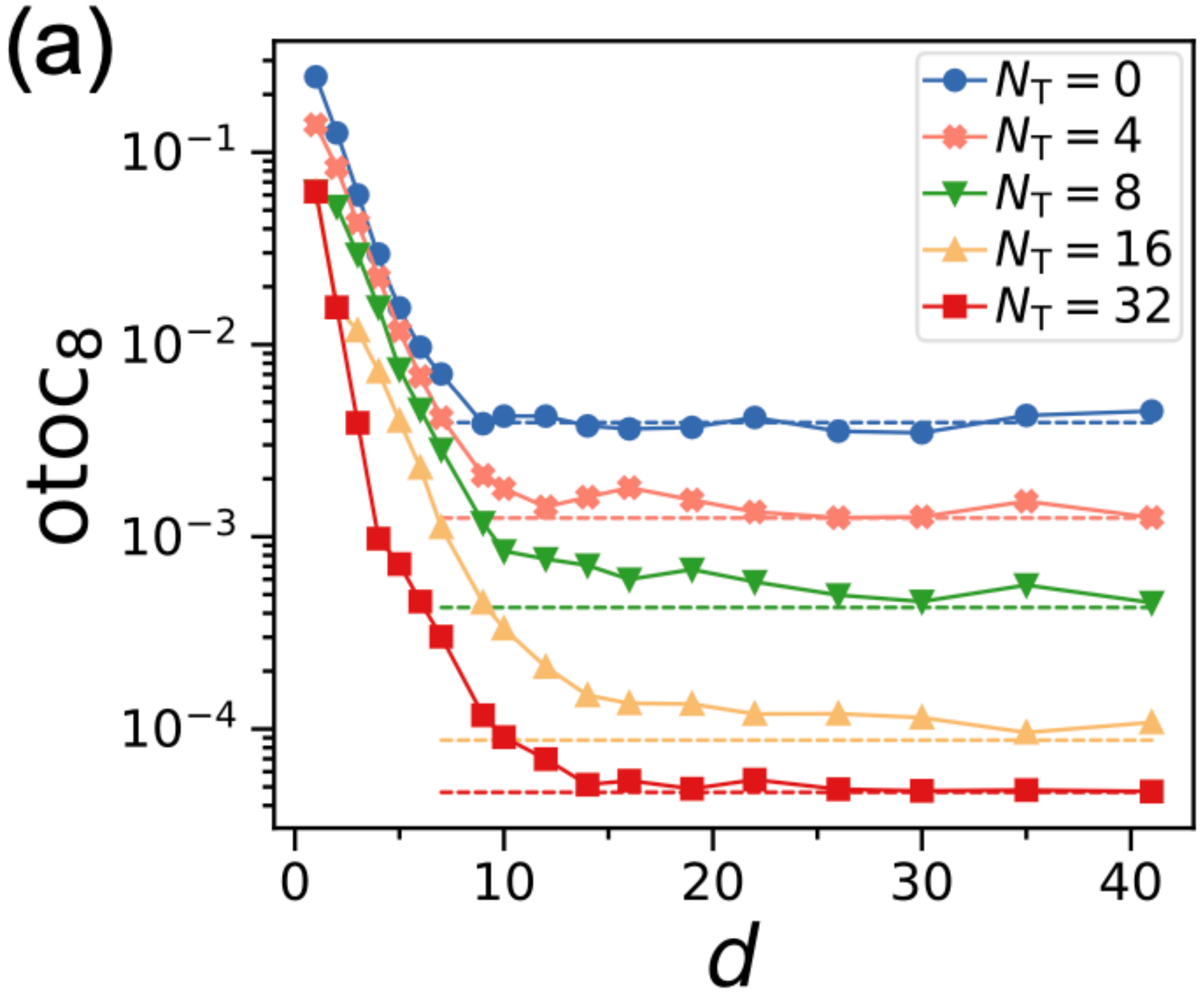
- Participation entropy[1]: $\mathcal{I}_q(|\psi\rangle) = \sum_k |\langle k|\psi\rangle|^{2q}$
  - Quantifies the spread of the wavefunction over basis states
- Multifractal flatness[2]: $\mathcal{F}(|\psi\rangle) = \mathcal{I}_3(|\psi\rangle) - \mathcal{I}_2^2(|\psi\rangle)$
- $\mathcal{F}(|\psi\rangle)$ averaged over $\mathscr{C}$[3]: $\bar{\mathcal{F}}(|\psi\rangle) = \mathop{\mathbb{E}}_{U_\mathrm{C}\in\mathcal{C}}[\mathcal{F}(U_\mathrm{C}|\psi\rangle)] = \dfrac{2(1-A_n(|\psi\rangle))}{(2^N+1)(2^N+2)}$
- —> Algorithm 2 allows us to efficiently measure $\bar{\mathcal{F}}(|\psi\rangle)$ without averaging

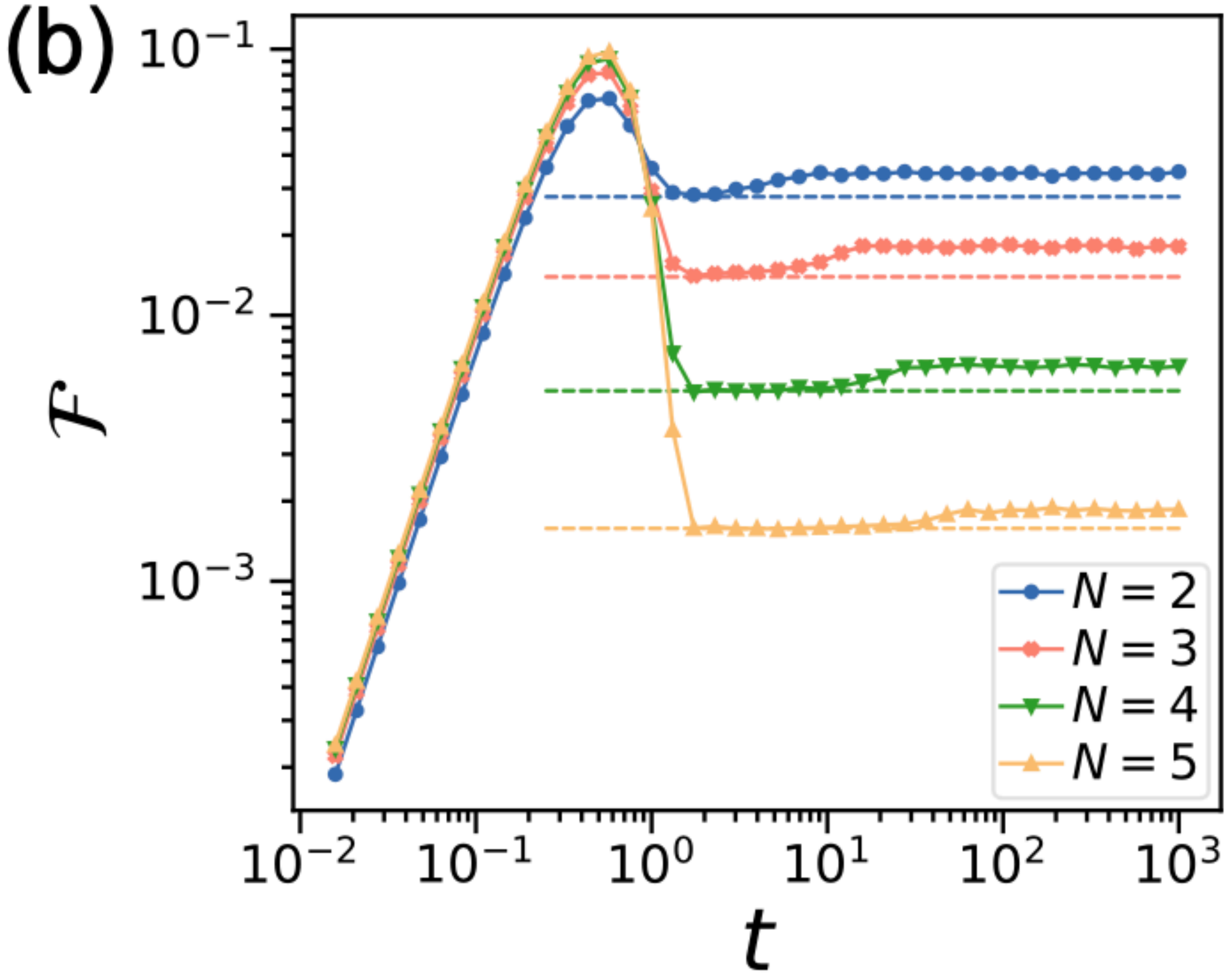
1.C Castellani and L Peliti, Journal of physics A: mathematical and general 19, L429 (1986)
2. P Sierant and X Turkeshi, Physical Review Letters 128, 130605 (2022)
3. X Turkeshi, M Schiro`, and Piotr Sierant,Phys. Rev. A 108, 042408 (2023)

# Scrambling




$$|\psi(t)\rangle = \exp(-iH_{\mathrm{GUE}}t)|0\rangle$$




$$\mathop{\mathbb{E}}_{U_C, U'_C \in \mathcal{C}_N}\left[\mathrm{otoc}_{4n}(U_C U U'_C, \sigma, \sigma')\right] = \frac{A_n(|U\rangle)4^N - 1}{(4^N - 1)^2}$$

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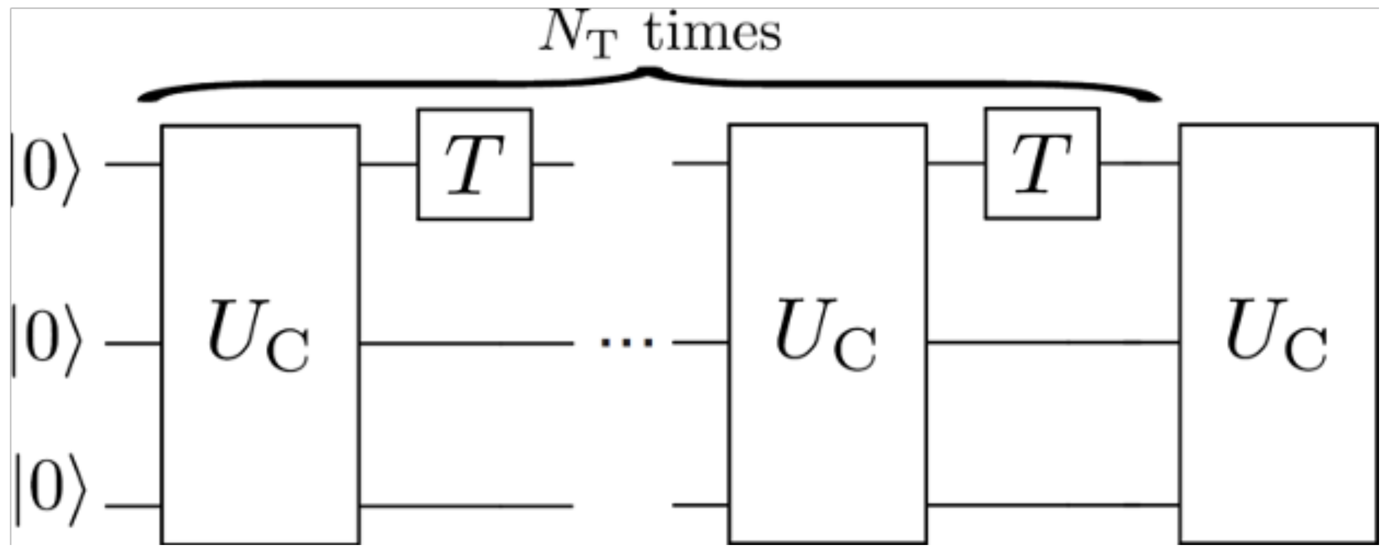


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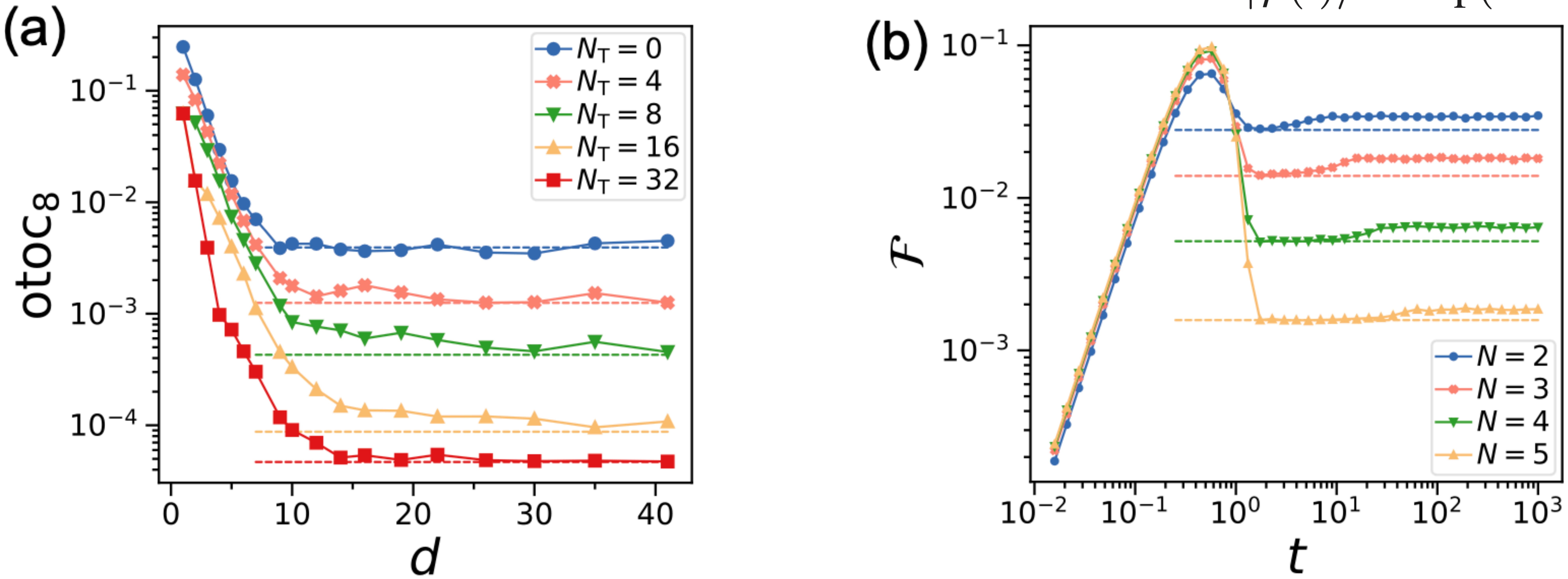


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# Summary

- **Efficiently measure SEs** with a **cost independent of qubit number** $N$
  - $->$ **Exponential improvement** over previous protocols [1,2]
- For integer $n > 1$, our protocol is **asymptotically optimal** with the number of copies scaling as $O(n\epsilon^{-2})$ and the classical post-processing time as $O(nN\epsilon^{-2})$
- Protocol **easy to implement** using Bell measurements
- Allows **efficient experimental characterisation** of different important properties of quantum states
- Demonstrated **efficient bound on nonstabilizerness monotones** which otherwise are difficult to compute beyond a few qubits

1. T. Haug and M.S. Kim, PRX Quantum 4, 010301 (2023)
2. S. F. E. Oliviero, L. Leone, A. Hamma, and S. Lloyd, npj Quantum Information 8, 148 (2022)

Technology Innovation Institute

# Efficient Quantum Algorithms for Stabilizer Entropies

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